Potential theory of Dirichlet forms with jump kernels blowing up at the boundary

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References

This talk is based on a joint project with Renming Song (University of Illinois, USA) and Zoran Vondraček (University of Zagreb, Croatia).

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[KSV] K, Song & Vondraček, Potential theory of Dirichlet forms with jump kernels blowing up at the boundary, arXiv:2208.09192[math.PR].

Outline

1. **Motivation by Two examples**
   - Example 1: Trace process
   - Example 2: Non-local Neumann problems

2. **Estimates of resurrection kernel in general case**

3. **Dirichlet forms with jump kernels blowing up at the boundary**
   - Setup
   - Decay rate of harmonic function

4. **Main results**

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Trace process

Let $d \geq 1$, $\mathbb{R}^d_+ := \{x = (\widetilde{x}, x_d) : x_d > 0\}$ and $\mathbb{R}^d_- := \{x = (\widetilde{x}, x_d) : x_d < 0\}$.

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$$\Delta^{\frac{\alpha}{2}} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) C_{d,\alpha} |x - y|^{-d-\alpha} \, dy,$$

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The process $Y = (Y_t)_{t \geq 0}$, defined by $Y_t = X_{\tau_t}$ called the *trace* process of X on $\mathbb{R}^d_+$.

Trace process through resurrection

The part of the trace process Y until its first hitting time of the boundary $\partial \mathbb{R}^d_+ = \{(\widetilde{x}, 0) : \widetilde{x} \in \mathbb{R}^{d-1}\}$ can be described in the following way:

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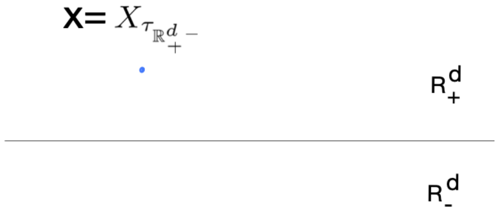

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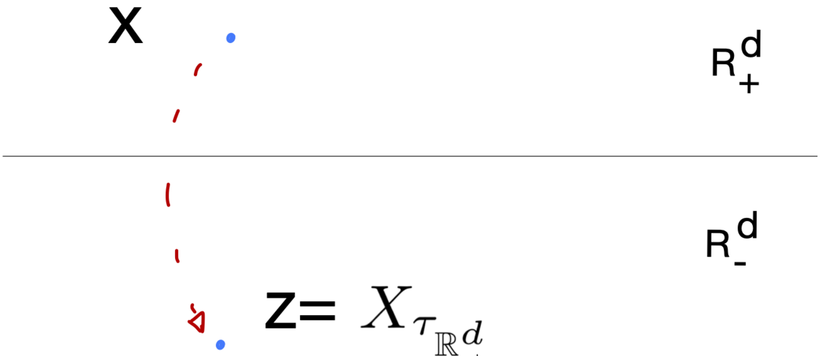

$$Z = X_{\tau_{\mathbb{R}^d_+}}$$

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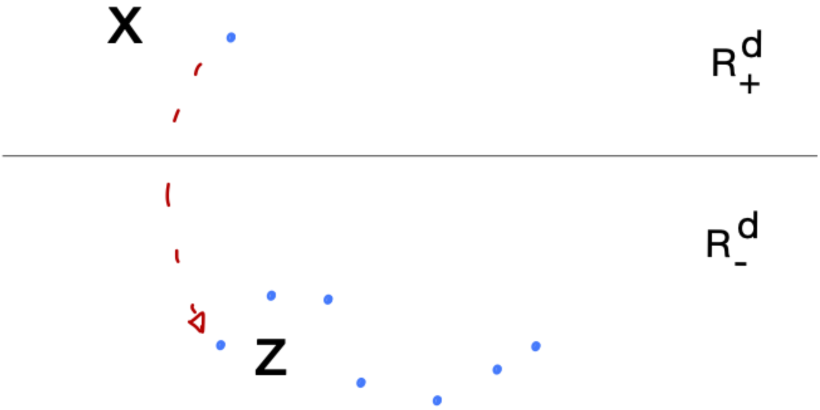

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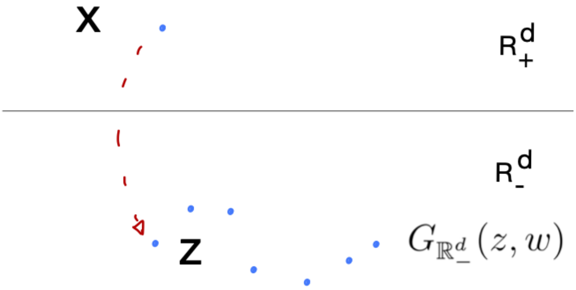

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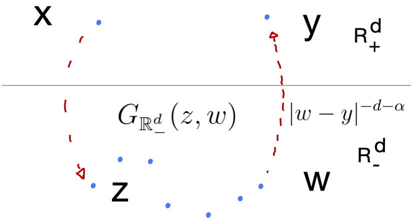

Trace process through resurrection

When X jumps out of $\mathbb{R}^d_+$ from the point x, we continue the process by resurrecting it at $y \in \mathbb{R}^d_+$ according to the kernel

$$q(x,y) := \int_{\mathbb{R}^d_-} |x-z|^{-d-\alpha} P_{\mathbb{R}^d_-}(z,y)\,dz, \quad x,y \in \mathbb{R}^d_+. \tag{1.1}$$

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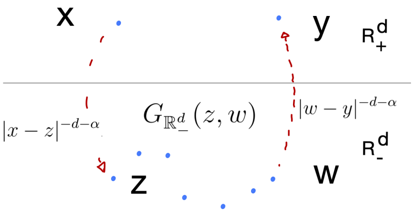

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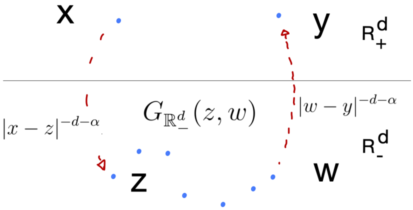

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Since $G^X_{\mathbb{R}^d_-}(\cdot,\cdot)$ is symmetric, it follows that $q(x,y) = q(y,x)$ for all $x,y \in \mathbb{R}^d_+$.

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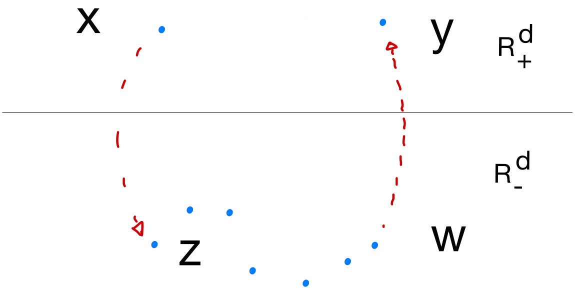

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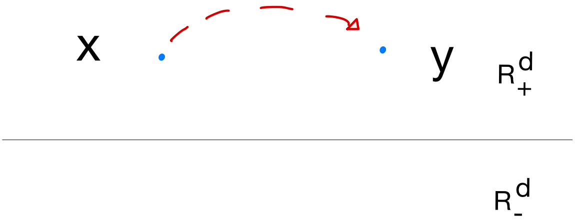

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$$J(x, y) \asymp q(x, y) \asymp |x - y|^{-d-\alpha} \left(\frac{|x - y|^2}{x_d \, y_d} \right)^{\alpha/2}, \quad x_d \wedge y_d \leq |x - y|.$$

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This asymptotic relation shows that the jump kernel $J(x,y)$ blows up with rate $x_d^{-\alpha/2}$ when x approaches the boundary $\partial\mathbb{R}^d_+$.

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Non-local Neumann problems with corresponding resurrection kernel

Another motivation is the process introduced in

- Dipierro, Ros-Oton & Valdinoci. Nonlocal problems with Neumann boundary conditions. Rev. Mat. Iberoam. 33 (2017), 377-416.
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For that process, the resurrection kernel $q(x, y)$ is given by (1.1) with the Poisson kernel $P_{\mathbb{R}^d_-}(z, y)$ replaced by $|z-y|^{-d-\alpha}/\int_{\mathbb{R}^d_+}|z-w|^{-d-\alpha}dw.$

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The jump kernel of this process blows up with rate $\log|x_d|$ when x approaches the boundary $\partial\mathbb{R}^d_+$.

Outline

Resurrection kernel in general case

We generalize these two examples by replacing the Poisson kernel $P_{\mathbb{R}^d_-}(z,y)$ and the kernel $|z-y|^{-d-\alpha}/\int_{\mathbb{R}^d_+}|z-w|^{-d-\alpha}dw$ by a very general return kernel $p(z,y)$.

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Let the kernel $y \mapsto p(z,y)$, $z \in \mathbb{R}^d_-$, $y \in \mathbb{R}^d_+$, be a probability density such that the corresponding resurrection kernel

$$q(x,y) = \int_{\mathbb{R}^d_-} |x-z|^{-d-\alpha} p(z,y)\, dy, \quad x,y \in \mathbb{R}^d_+,$$

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This flexibility in choosing the return kernel allows us to obtain resurrection kernels with various blow-up rates at the boundary.

A general upper bound

$$q(x,y)\le c(x_d\wedge y_d)^{-d-\alpha}$$

An example of general forms

We consider the following general resurrection kernel:

$$q(x,y) := c\int_{\mathbb{R}^d_-} \Psi\left(\frac{|y-z|^2}{y_d|z_d|}\right)\frac{|z_d|^\alpha}{|y-z|^{d+\alpha}}\frac{dz}{|x-z|^{d+\alpha}}, \quad x,y\in\mathbb{R}^d_+,$$

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Moreover,

$$q(x,y) = q(y,x).$$

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Exmaple

(a) For the trace process of an isotopic α-stable process on $\mathbb{R}^d_+$,

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$p(z,y) = |z-y|^{-d-\alpha}/\mu(z)$ where $\mu(z) = \int_{\mathbb{R}^d_+}|z-y|^{-d-\alpha}dy$. Since $\mu(z) = c^{-1}|z_d|^{-\alpha}$, we get that

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Let

$$\mathcal{B}(x,y) := 1 + q(x,y)|x-y|^{d+\alpha}$$

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In [KSV] , we have established estimates of resurrection kernel for general Ψ with weak scaling conditions.
As a corollary of this general result, we have the following.

Blow-up at the boundary: example

Corollary: Let $\gamma \in (-\infty, 1 \wedge \alpha)$ and $\delta \in \mathbb{R}$. Suppose $\Psi(t) = t^{\gamma} \log^{\delta} t$, $t \geq 2$,

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$$q(x,y) \asymp J(x,y) \asymp |x-y|^{-d-\alpha} \begin{cases} \left(\frac{|x-y|^2}{x_d y_d}\right)^\gamma \log^\delta \left(\frac{|x-y|^2}{x_d y_d}\right) & \text{when } \gamma > 0; \\ \log^{\delta+1} \left(\frac{|x-y|^2}{x_d y_d}\right) & \text{when } \delta > -1, \gamma = 0; \\ \log\left(e + \log\left(\frac{|x-y|^2}{x_d y_d}\right)\right) & \text{when } \delta = -1, \gamma = 0; \\ 1 & \text{when } \delta < -1, \gamma = 0 \text{ or } \gamma < 0. \end{cases}$$

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Let $d \geq 1$, $\alpha \in (0,2)$ and assume that $0 \leq \beta_1 \leq \beta_2 < 1 \wedge \alpha$.

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Let Φ be a positive function on $(0,\infty)$ satisfying $\Phi \equiv 1$ on $(0,2)$ and the following weak scaling condition on $[2,\infty)$: There exist constants $C_1, C_2 > 0$ such that

$$C_1(R/r)^{\beta_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq C_2(R/r)^{\beta_2}, \quad 2 \leq r < R < \infty.$$

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$$|\mathcal{B}(x,y) - \mathcal{B}(x,x)| \leq C \left(\frac{|x-y|}{x_d \wedge y_d} \right)^{\theta} \qquad \text{for all } x, y \in \mathbb{R}^d_+ \text{ with } x_d \wedge y_d \geq a|x-y|.$$

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In case $d \geq 2$, $\mathcal{B}(x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = \mathcal{B}(x,y)$ for all $x, y \in \mathbb{R}^d_+$ and $\widetilde{z} \in \mathbb{R}^{d-1}$.

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Remark: (1) **(A3)** $\Rightarrow\mathcal{B}(x,y)\ge c_1>0$. (2) **(A4)** $\Rightarrow\mathcal{B}(x,x)\equiv c_2>0$.

Dirichlet form with critical killing (Potential) term

For $\kappa \in [0, \infty)$, set

$$\mathcal{E}^{\kappa}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)dydx + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa x_d^{-\alpha}dx.$$

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Let $\mathcal{F}^0$ be the closure of $C_c^\infty(\mathbb{R}^d_+)$ in $L^2(\mathbb{R}^d_+, dx)$ under $\mathcal{E}_1^0 := \mathcal{E}^0 + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+, dx)}$.
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Under assumptions **(A1)**-**(A4)**, for all $\kappa \in [0, \infty)$, there exists a symmetric, scale invariant and horizontally translation invariant Hunt process $Y^\kappa = ((Y_t^\kappa)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d_+})$ associated with $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$.

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Decay rate and the constant on the killing function

We now associate the constant κ from the killing function $x_d^{-\alpha}$ with a positive parameter $p = p_\kappa > 0$ which will be the decay rate of harmonic functions.

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Then $C(\alpha, 0, \mathcal{B}) = C(\alpha, \alpha - 1, \mathcal{B}) = 0$ and the function $q \mapsto C(\alpha, q, \mathcal{B})$ is strictly increasing and continuous on $[(\alpha - 1)_+, \alpha - \beta_2)$.

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When $\Phi(r) = r^\beta$ with $\beta \in (0, 1 \wedge \alpha)$, it holds that $\lim_{q \uparrow \alpha - \beta} C(\alpha, q, \mathcal{B}) = \infty$, so $\kappa \mapsto p_\kappa$ is an increasing bijection from $[0, \infty)$ onto $[(\alpha - 1)_+, \alpha - \beta)$.

In the remainder of this talk we will fix $\kappa \in [0, \lim_{q \uparrow \alpha - \beta_2} C(\alpha, q, \mathcal{B}))$, and assume $\alpha > 1$ if $\kappa = 0$ so that $p_0 = \alpha - 1 > 0$.

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$$L^{\mathcal{B}} f(x) = L^{\mathcal{B},\kappa} f(x) = \text{p.v.} \int_{\mathbb{R}^d_+} (f(y) - f(x)) J(x,y)\, dy - C(\alpha, p, \mathcal{B}) x_d^{-\alpha} f(x),\ x \in \mathbb{R}^d_+,$$

whenever the principal value integral makes sense.

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Hence the operator $L^{\mathcal{B}}$ annihilates the p-th power of the distance to the boundary.

Outline

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Assume that $\mathcal{B}$ satisfies **(A1)**-**(A4)** and $\kappa = C(\alpha, p, \mathcal{B})$. Then there exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $B((\widetilde{w}, 0), 2r) \cap \mathbb{R}^d_+$ with respect to Y^κ and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

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$$\frac{f(x)}{x_d^p} \le C \frac{f(y)}{y_d^p}, \quad x, y \in B((\widetilde{w}, 0), r/2)) \cap \mathbb{R}^d_+.$$

Sharp two-sided estimates for the Green function

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$$G^\kappa(x,y) \asymp \begin{cases} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p \frac{1}{|x-y|^{d-\alpha}}, & \alpha < d; \\ \left(\frac{x \wedge y}{|x-y|} \wedge 1\right)^p \log\left(e + \frac{x \vee y}{|x-y|}\right), & \alpha = 1 = d; \\ \left(\frac{x \wedge y}{|x-y|} \wedge 1\right)^p (x \vee y \vee |x-y|)^{\alpha - 1}, & \alpha > 1 = d. \end{cases}$$

The results in this talk can be considered as a counterpart of our previous work on jump kernels vanishing at the boundary:

[KSV1] K, Song & Vondraček: On potential theory of Markov processes with jump kernels decaying at the boundary. *Potential Analysis* (2023).

[KSV2] K, Song & Vondraček, Sharp two-sided Green function estimates for Dirichlet forms degenerate at the boundary. To appear in *Journal of the European Mathematical Society (JEMS)*, 2023.

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The research in [KSV1]-[KSV3] was motivated by the fact that subordinate killed Lévy processes have jump kernels vanishing at the boundary.

Combined with jump kernel vanishing at the boundary

As a particular case,

$$\mathcal{B}(x,y) \asymp \left(\frac{x_d}{|x-y|} \wedge 1\right)^{\beta} \left(\frac{y_d}{|x-y|} \wedge 1\right)^{\beta} \asymp \left(\frac{x_d y_d}{|x-y|^2} \wedge 1\right)^{\beta} = \widetilde{B}(x,y).$$

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In the decay case with $\mathcal{B}$ as above, we have already proved the BHP and the Green function estimate with the decay rate equal to the p-th power of the distance to the boundary. Thus, we now have these results for all $-(1 \wedge \alpha) < \beta < \infty$.

No killing case

Suppose $\kappa = 0$, $\alpha > 1$, **(A2)** and **(A4)** hold and

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The corresponding non-local operator is

$$Lf(x) := \text{p.v.} \int_{\mathbb{R}^d_+} (f(y) - f(x)) \frac{\mathcal{B}(x,y)}{|x-y|^{d+\alpha}}\, dy, \quad x \in \mathbb{R}^d_+ ,$$

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This covers the case of the censored α-stable process ($\mathcal{B}(x,y) \equiv 1$), $\alpha \in (1,2)$, studied in Bogdan, Burzdy & Chen, 03, in the half-space case.

Thank you.

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